Question 3

(i) Use of $\dfrac{1}{5+\sqrt{24}} = 5-\sqrt{24}$ (by the difference of two squares) **B1**

Then $= \left(5+\sqrt{24}\right)^4 + \dfrac{1}{\left(5+\sqrt{24}\right)^4}$ $= = \left(5+\sqrt{24}\right)^4 + \left(5-\sqrt{24}\right)^4$

Use of the binomial theorem (e.g.) to expand at least one of these **M1**

$\left(5+\sqrt{24}\right)^4 = 5^4 + 4 \,.\, 5^3 \,.\, \sqrt{24} + 6.\, 5^2 \,.\, 24 + 4 \,.\, 5 \,.\, 24\sqrt{24} + 24^2$ **A1**

$\left(5-\sqrt{24}\right)^4 = 5^4 - 4 \,.\, 5^3 \,.\, \sqrt{24} + 6.\, 5^2 \,.\, 24 - 4 \,.\, 5 \,.\, 24\sqrt{24} + 24^2$ **B1** FT

Adding to get a number (or series of numbers) with no $\sqrt{24}$'s

$\left(5+\sqrt{24}\right)^4 + \left(5-\sqrt{24}\right)^4 = 2\{5^4 + 6 \,.\, 5^2 \,.\, 24 + 24^2\} = 2(625 + 144 \times 25 + 576)$

$= 1250 + 72 \times 100 + 1152 = 9602$

A1 or **B1** for convincing explanation that the result must be an integer

OR

$$\left(5+\sqrt{24}\right)^4 + \frac{1}{\left(5+\sqrt{24}\right)^4} = \left[\left(5+\sqrt{24}\right)^2 + \frac{1}{\left(5+\sqrt{24}\right)^2}\right]^2 - 2$$ **M1** **A1**

$$= \left[\left\{\left(5+\sqrt{24}\right) + \frac{1}{\left(5+\sqrt{24}\right)}\right\}^2 - 2\right]^2 - 2$$ **M1** **A1**

$$= \left[\left\{\left(5+\sqrt{24}\right) + 5 - \sqrt{24}\right\}^2 - 2\right]^2 - 2 = (10^2 - 2)^2 - 2 = 98^2 - 2 = 9602$$

A1 or **B1** for convincing explanation that the result must be an integer **5**

Now $5+\sqrt{24} < 5 + 5 = 10$ **B1** and $5+\sqrt{24} > 5+\sqrt{20.25} = 5 + 4.5 = \dfrac{19}{2}$ **B1**

Thus $\dfrac{19}{2} < 5+\sqrt{24} < 10 \Rightarrow \dfrac{2}{19} > \dfrac{1}{5+\sqrt{24}} > \dfrac{1}{10}$

Also, $\dfrac{2}{19} < \dfrac{11}{100}$ since $2 \times 100 < 11 \times 19$ $(200 < 209)$ **B1**

so that $\dfrac{1}{10} < \dfrac{1}{5+\sqrt{24}} < \dfrac{2}{19} < \dfrac{11}{100}$ i.e. $0.1 < \dfrac{1}{5+\sqrt{24}} < \dfrac{2}{19} < 0.11$ **3**

Then $0.1^4 < \left(5-\sqrt{24}\right)^4 < 0.11^4 \Rightarrow 0.000\,1 < \left(5-\sqrt{24}\right)^4 < 0.000\,146\,41$

$\Rightarrow -0.000\,1 > -\left(5-\sqrt{24}\right)^4 > -0.000\,146\,41$

$\Rightarrow 9601.999\,9 > 9602 - \left(5-\sqrt{24}\right)^4 = \left(5+\sqrt{24}\right)^4 > 9601.999\,85$

M1 Complete method (using both sides of the inequality) **A1** correct number work

and $\left(5+\sqrt{24}\right)^4 = 9601.9999$ to 4 d.p. **A1** correct and completely convincing **3**

(ii) $\left(N+\sqrt{N^2-1}\right)^k + \dfrac{1}{\left(N+\sqrt{N^2-1}\right)^k} = \left(N+\sqrt{N^2-1}\right)^k + \left(N-\sqrt{N^2-1}\right)^k = M$ (an integer)

M1 **A1** **B1**

For the B1, they must explain why this expression is an integer (e.g. by the binomial theorem)

Now $\left(N-\sqrt{N^2-1}\right)^k$ is positive, and the reciprocal of a number > 1 **B1**

so $\left(N-\sqrt{N^2-1}\right)^k \to 0+$ as $k \to \infty$ **B1**

Also, $2N-\frac{1}{2} < N+\sqrt{N^2-1} < 2N \Rightarrow \dfrac{1}{2N-\frac{1}{2}} > N-\sqrt{N^2-1} > \dfrac{1}{2N}$ **M1** **A1**

Thus $\left(N+\sqrt{N^2-1}\right)^k = M - \left(N-\sqrt{N^2-1}\right)^k$ differs from an integer (M) **M1**

by less than $\left(\dfrac{1}{2N-\frac{1}{2}}\right)^k = (2N-\frac{1}{2})^{-k}$ **A1** **9**